Toulouse School of Economics

M1 INTERMEDIATE ECONOMETRICS

Large-sample inference

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Setting

We now consider the case where

$$Y = X'\beta + e, \qquad \mathbb{E}(Xe) = 0.$$

Here, the exact distribution of the least-squares estimator is unknown.

We rely on asymptotics, using that

$$\sqrt{n}(\hat{\beta} - \beta) \underset{d}{\rightarrow} N(0, V_\beta),$$

and

$$\hat{V}_\beta \underset{d}{\rightarrow} V_\beta$$

as $n \rightarrow \infty$.

The developments to test a collection of m linear contrasts $\theta = R'\beta$ are essentially as before.

It is immediate that

$$\sqrt{n}(\hat{\theta} - \theta) \underset{d}{\rightarrow} N(0, V_\theta),$$

and that

$$\hat{V}_\theta = R'\hat{V}_\beta R \underset{p}{\rightarrow} V_\theta = R'V_\beta R$$

as $n \rightarrow \infty$.

We can use these results to validate the same test procedures as before.

The key difference to before is that, now, size and power can only be approximated.

Test procedure

Under the null $\mathbb{H}_0 : \theta = \theta_0$,

$$W = n(\hat{\theta} - \theta_0)' \hat{V}_\theta^{-1} (\hat{\theta} - \theta_0) \underset{d}{\rightarrow} \chi^2_m,$$

and so we take critical values c_α to be the $(1-\alpha)$th quantile of the χ^2_m distribution.

This ensures that

$$\lim_{n \to \infty} \mathbb{P}_{\theta_0}(W > c_\alpha) = \alpha.$$

So, in practice, there is a difference between the theoretical/asymptotic size of the test and its actual size.

Notice that using V_θ in place of $\hat{V}_\theta$ does not affect any of the results on this slide.

Consistency

Consistency means that the probability of making a type-II error goes to zero as $n \to \infty$.

The above procedure is consistent against all fixed alternatives.

To see this, let

$$b_\theta = V_\theta^{-1/2}(\theta - \theta_0).$$

Then

$$\sqrt{n}\hat{V}_\theta^{-1/2}(\hat{\theta} - \theta_0) \underset{d}{\to} \sqrt{n}V_\theta^{-1/2}(\hat{\theta} - \theta) + \sqrt{n}\, b_\theta \underset{d}{\to} N(0, I_m) + \sqrt{n}\, b_\theta.$$

Therefore,

$$W \underset{d}{\rightarrow} \chi^2_m(n\, b'_\theta b_\theta).$$

As $n \to \infty$ the non-centrality parameter $n\, b'_\theta b_\theta > 0$ diverges to infinity.

Consistency follows:

$$\lim_{n \to \infty} \mathbb{P}_\theta(W > c_\alpha) = 1$$

for any fixed value θ (different from θ_0).

Local alternatives

Consistency is desirable.

Does not say much about what power we can expect for a given sample size and a given alternative.

Consider shrinking alternatives of the form

$$\theta = \theta_0 + \frac{h}{\sqrt{n}}$$

for some fixed value h.

Makes the deviation smaller as the sample size grows in a way that

$$\sqrt{n}(\theta - \theta_0) = h$$

remains fixed.

It should be clear where the rate of $\sqrt{n}$ comes from.

Now

$$\sqrt{n}\hat{V}_\theta^{-1/2}(\hat{\theta} - \theta_0) = \sqrt{n}V_\theta^{-1/2}(\hat{\theta} - \theta) + \sqrt{n}\, b_\theta \underset{d}{\rightarrow} N(0, I_m) + V_\theta^{-1/2}h$$

and the bias term does not blow up as n grows.

This means that the non-centrality parameter does not diverge. Now,

$$\mathbb{P}_\theta(W > c_\alpha) \rightarrow 1 - \mathcal{X}_m(c_\alpha, h' V_\theta^{-1} h).$$

for $\mathcal{X}_m$ the CDF of the relevant non-central Chi-squared distribution.

We then substitute back in $h = \sqrt{n}(\theta - \theta_0)$ and use the probability

$$1 - \mathcal{X}_m(c_\alpha, n\,(\theta - \theta_0)' V_\theta^{-1}(\theta - \theta_0))$$

to approximate the power against alternative θ for a sample of size n.

Nonlinear hypothesis

Once we rely on an asymptotic approximation we are no longer bound to test only linear hypotheses.

Let $r : \mathbb{R}^k \to \mathbb{R}^m$ be a vector valued function mapping β to a set of m nonlinear transformations.

Then we may wish to test the null $\mathbb{H}_0 : \theta = \theta_0$ for $\theta = r(\beta)$.

The delta method implies that

$$\sqrt{n}(\hat{\theta} - \theta) \underset{d}{\to} N(0, V_\theta)$$

for

$$V_\theta = R(\beta)' V_\beta R(\beta)$$

where

$$R(b) = \frac{\partial r'(b)}{\partial b}.$$

is the $k \times m$ Jacobian matrix of the vector function r.

Delta method

Mean-value expansion of $r(\hat{\beta})$ around β gives

$$r(\hat{\beta}) = r(\beta) + R(\beta^*)'(\hat{\beta} - \beta)$$

for some $\beta^* \underset{p}{\rightarrow} \beta$.

If the Jacobian is continuous at β, $R(\beta^*) \underset{p}{\rightarrow} R(\beta)$ and so

$$\sqrt{n}(r(\hat{\beta}) - r(\beta)) \underset{d}{\rightarrow} R(\beta)'\sqrt{n}(\hat{\beta} - \beta).$$

Because $\sqrt{n}(\hat{\beta} - \beta) \underset{d}{\rightarrow} N(0, V_\beta)$,

$$\sqrt{n}(r(\hat{\beta}) - r(\beta)) \underset{d}{\rightarrow} N(0, R(\beta)' V_\beta R(\beta))$$

follows.

Test statistic

We now use

$$W = n\,(\hat{\theta} - \theta_0)'\hat{V}_{\theta}^{-1}(\hat{\theta} - \theta_0)$$

and proceed in the same way as before.

The variance estimator is $\hat{V}_{\theta} = R(\hat{\beta})'\hat{V}_{\beta}R(\hat{\beta})$ and is readily shown to be consistent for V_{θ}.